"HEAT RESERVOIR" BOUNDARY CONDITION AS LIMITING RELATION

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Asymptotic analysis provides a basis for deriving the "heat reservoir" boundary condition when a solid surface is in contact with a region of vigorous liquid motion; it simplifies the formulation of multilayer heat and mass transfer problems under definite conditions. An effective heat conduction equation resembling Taylor's dispersion equation is obtained for the heat transfer in the liquid.

Introduction. The heat reservoir boundary condition appears naturally in considering certain heat and mass transfer problems [1-4]. Its distinctive feature is the fact that the boundary ratio contains the time derivative of the function sought. This condition is traditionally just postulated, as in the aforementioned studies; some considerations are advanced regarding the rapid temperature (concentration) equalization within a region that is arbitrarily termed a heat reservoir, and a heat (material) balance equation is written for it. This approach leaves a number of questions unclear: how is the concept of "rapidly equalized" to be evaluated numerically and what refinements of this condition are possible when the temperature does not equalize with sufficient rapidity within the reservoir? Here we can clearly see that there is some dimensionless parameter whose limiting value corresponds to the idealization represented in the process description by the boundary condition under consideration. Methodology of this sort was implemented in [5, 6], where the formulation of the heat reservoir condition was investigated for contact between two bodies with substantially different coefficients of thermal conductivity. We will examine the two-dimensional problem involving thermal contact between a solid and a region occupied by a liquid undergoing rapid displacement. The coefficients of thermal conductivity in the solid and reservoir will be considered to be commensurate with contact taking place over only a portion of the solid surface (see Fig. 1).

Formulation of the Problem. The heat conduction equations for non-one-dimensional liquid flow within a reservoir, which have convective terms that depend on the aggregate of the spatial coordinates, are quite complicated for analysis by exact methods. We will somewhat simplify the problem by assuming the liquid flow to be two-dimensional and the liquid itself to be incompressible. The convection field will be assumed to be known and expressed in terms of the flow function Ψ, which exists under our conditions. We will ascertain below that only a definite integral characteristic of the convection field is actually required. The liquid flow can be induced by various fluid dynamic factors. This is not important for our analysis.

The reservoir will be treated as elongated in the direction normal to the solid (see Fig. 1). The basic equations of the problem are:

$$u = \partial\Psi/\partial y,\ v = -\partial\Psi/\partial x \tag{1}$$

which are the velocity component ratios following from the conditions that the liquid be incompressible and the flow be two-dimensional;

$$\varepsilon^2 \frac{\partial T_*}{\partial \zeta} + \varepsilon \operatorname{Pe}\left(u \frac{\partial T_*}{\partial x} + v \frac{\partial T_*}{\partial y}\right) = \frac{\partial^2 T_*}{\partial y^2} + \varepsilon^2 \frac{\partial^2 T_*}{\partial x^2} \tag{2}$$

which is the equation for the convective heat conduction in the reservoir, and

$$\frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2},\quad \tau = \frac{\lambda t}{\rho c}, \tag{3}$$

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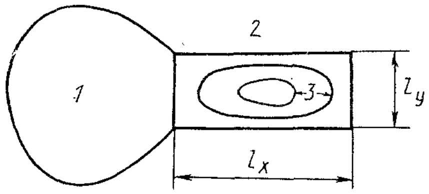

Heat transfer diagram: 1) Solid; 2) Liquid; 3) Current lines.

which is the heat transfer conduction equation for the primary body. It is necessary to supplement the latter equation by formulating the "heat reservoir" condition at the reservoir boundary. This is done in order to simplify the heat transfer problem in a two-layer region with different process parameters and heat transfer mechanisms, and it entails replacing the influence of the region in which the complex convective heat transfer takes place with an asymptotically valid boundary condition. Here there is no need for detailed solution of the full adjoint heat transfer problem with the conditions on the contact surface:

$$T|_{X=0} = T_*|_{X=0},\ \lambda \left.\frac{\partial T}{\partial X}\right|_{X=0} = \lambda_* \left.\frac{\partial T_*}{\partial X}\right|_{X=0}, \tag{4}$$

which are required in order to complete the formulation of the problem for Eqs. (2)-(3). We do not have to have any initial conditions at this stage of the analysis. It should be remarked that the heat liberated in the liquid as a result of viscous energy dissipation is neglected. The length scales parallel to the x and y axes in dimensionless equation (2) are taken to be the reservoir dimensions ℓ_x and ℓ_y respectively, and their connection to the dimensional parameters is given by the formulas:

$$\zeta = \lambda_* t/\rho_* c_* l_x^2,\ \mathrm{Pe} = \Psi_0 \rho_* c_*/\lambda_*,\ \varepsilon = l_y/l_x \ll 1, \tag{5}$$

where Ψ_0 is the flow function scale. Quantities pertaining to the reservoir carry the index *.

We will consider the region occupied by the reservoir to be rectangular in shape, in order not to make our exposition too unwieldy. Possible complications of reservoir shape will be discussed below.

1. Derivation of Heat Dispersion Equation. Our (intermediate) goal at this point is to obtain a simpler one-dimensional equation in place of (2). Besides being of aid in constructing a boundary condition of the heat reservoir type, this transformation is itself of interest for description of heat and mass transfer in liquid media in the presence of complex convection currents.

Since the parameter ε is a small quantity, we will seek the solution of the problem in the form of a perturbation-method series:

$$T_* = T_*^0 + \varepsilon T_*^1 + \varepsilon^2 T_*^2 + \ldots. \tag{6}$$

The boundary conditions on the variable x are not significant for this stage of our analysis, and Eq. (2) is therefore subject only to the following conditions:

$$\partial T_*/\partial y|_{y=0;1} = 0, \tag{7}$$

which expresses absence of a heat flux at the corresponding reservoir region boundaries, and the stream function will be assumed to vanish at the flow region boundary.

It must be noted that the order of the terms we have chosen in Eq. (2) is intended to allow them to be successively taken into account, as in Taylor's analysis [7] of the dispersion of an impurity in a round pipe. We are essentially proposing another approach to derivation of the dispersion equation, one that makes it possible to analyze material or energy transport with more complex velocity profiles than those realized in prismatic pipes.

The proposed small-parameter procedure was implemented in [8-10]; the problem in [9] was analogous to that being considered here, i.e., Eq. (1), (2), and (7). We will therefore only give the final result:

$$\frac{\partial T_*^0}{\partial \zeta} = \frac{\partial}{\partial x}\left\{[1 + \mathrm{Pe}^2 a_*(x)] \frac{\partial T_*^0}{\partial x}\right\} \tag{8}$$

which is the equation we are seeking for the effective heat conduction in the reservoir. Here

$$a_*(x) = \int_0^1 \Psi^2(x, y)\, dy \tag{9}$$

is the coefficient of heat dispersion and originates in the nonuniformity of the rate of heat transfer by convective motion under the influence of molecular heat conduction. The superior zero in Eq. (8) will henceforth be omitted for brevity, and this equation will be employed to describe the heat transfer in the reservoir.

When liquid motion in the reservoir is vigorous ($\mathrm{Pe} \to \infty$), the dispersion coefficient substantially exceeds the coefficient of molecular thermal diffusivity, which leads to rapid temperature equilization over the reservoir volume; however, no matter how large the dispersion coefficient within the region, $a_*(x) \to 0$ at its boundary, there should consequently be a transitional (boundary-layer) zone where the temperature changes more abruptly. The thickness of the boundary layer δ is easily estimated if we note that the function $a_*(x)$ has a fourth-order zero when $x = 0$ (since the stream function and its normal derivative vanish at the boundary $x = 0$) and that the dispersion and thermal diffusivity coefficients in this layer are of the same order of magnitude. We have $\delta/l_x = O(1/\sqrt{\mathrm{Pe}})$, where $\mathrm{Pe} \to \infty$. We introduce the notation $\varkappa = a_*^{(4)}(0)/4!$, carry our a similar deformation of the coordinate, i.e., $z = x(\mathrm{Pe}^2\varkappa)^{1/4}$, and wishing to obtain a nonsteady state equation approximating the boundary layer, come up against the need to "stretch" the time: $\theta = \zeta(\varkappa \mathrm{Pe})^{1/2}$. As a result, we have the limiting equation

$$\frac{\partial T_*}{\partial \theta} = \frac{\partial}{\partial z}\left[(1 + z^4)\frac{\partial T_*}{\partial z}\right], \tag{10}$$

which can be used to analyze various heat and mass transfer problems (and not just for the purposes of the present study; see [10]).

The need to consider the limiting variant $\mathrm{Pe} \to \infty$ in deriving a condition of the "heat reservoir" type also follows from comparison of the characteristic heating times in the solid ($t_1 = \rho c \ell_x^2/\lambda$) and the reservoir ($t'_2 = \ell_x^2/\Psi_0$, as a consequence of convection). With the "guideline" criteria generally utilized [1-4] in constructing the "heat reservoir" condition, it is required that $t'_2 \ll t_1$, whence it follows the $\mathrm{Pe} \gg 1$.

2. Derivation of Condition of "Heat Reservoir" Type. The processes that occur during the characteristic time for the solid, $t_1 = \rho c \ell_x^2/\lambda$, are of primary interest in our problem of deriving a boundary condition of the "heat reservoir" type. It should be noted that the time scale for Eq. (10) to be "operative" substantially exceeds the characteristic time of basic equation (8) when $\mathrm{Pe} \to \infty$, so that we can henceforth utilize (10) in the quasi-steady-state approximation for analyzing processes with a scale $t_2 = \rho_* c_* \ell_x^2/\lambda_*$, which we will assume to be similar in order of magnitude to t_1. We would otherwise arrive at the variants considered previously [5, 6]. When $t_2 = O(t_1)$, the main feature distinguishing the case under investigation from those in [5, 6] is the fact that the effective coefficient of thermal diffusivity $\mathrm{Pe}^2 a_*(x)$ ($\mathrm{Pe} \to \infty$) in Eq. (8) is greater than the corresponding coefficient a in (3) over virtually the entire reservoir volume but not near the contact boundary, where $a_*(x) \to 0$. The other parameters are comparable in magnitude. As a consequence, the results of [5, 6] cannot be directly extended to our example.

The specific problem under consideration thus permits use of simple solutions of Eq. (10), by virtue of the applicability of the quasi-steady-state approximation. However, this question is also helpful in finding the corrections to the basic approximation of the "heat reservoir" condition, i.e., we have formulated equations whose analysis enables us to approach derivation of this condition by means of asymptotic expansions rather than on the basis of preliminary estimates, in which case one can only hope to obtain the primary approximation.

We now turn to the principal objective of the present study, i.e., construction of a condition of the "heat reservoir" type. Switching from Eq. (2) to (8) does not alter the structure of conditions (4), as a consequence of the equality $a_*(0) = 0$, but it must be taken into account that (8) is essentially averaged over the y coordinate [8-10]. When we integrate Eq. (8) over x, taking into account the lack of any heat flux at the outer reservoir boundary, i.e., $\partial T_*/\partial X|_{X=1} = 0$, we find

$$\frac{\partial \langle T_* \rangle}{\partial \zeta} = -\left.\frac{\partial T_*}{\partial x}\right|_{x=0}, \quad \langle T_* \rangle = \int_0^1 T_* dx. \tag{11}$$

By virtue of our earlier remarks, the quantity $\langle T_* \rangle$ can serve as the boundary condition at infinity for boundary equation (10), since the value of the function T_* in the integral in (11) is determined almost everywhere [except in a zone of thickness $\delta/\ell_X = O(Pe^{-1/2})$] by the exterior solution [11, 12], which depends solely on ζ as can easily be seen.

We will show that $\langle T_* \rangle$ in [11] can, in first approximation for Pe^{-1}, validly be replaced by the average temperature of the reservoir-system interface. Actually, if we take into account the quasi-steady-state character of Eq. (10) when $\zeta = O(1)$, we can easily find its integral and then the relationship in which we are interested:

$$\langle T_* \rangle - \bar{T}|_{x=0} = (\varkappa \, \mathrm{Pe}^2)^{-1/4} \left.\frac{\partial \bar{T}}{\partial x}\right|_{x=0} \int_0^\infty \frac{dx}{1+x^4} = O(\mathrm{Pe}^{-1/2}) \left.\frac{\partial \bar{T}}{\partial x}\right|_{x=0}, \tag{12}$$

where, since T_* was averaged over y in converting to (8) [8-10], averaging is also carried out in boundary conditions (4):

$$\bar{T}|_{x=0} = \int_0^\infty T|_{x=0}\, dy, \quad \left.\frac{\partial \bar{T}}{\partial x}\right|_{x=0} = \int_0^1 \left.\frac{\partial T}{\partial x}\right|_{x=0} dy.$$

It follows from (12) that $\langle T_* \rangle = \bar{T}|_{X=0}$ up to terms of order $O(Pe^{-1/2})$. Substitution of the latter relation into (11) gives us the heat reservoir condition sought:

$$\rho_* c_* l_x \partial \bar{T}/\partial t|_{X=0} = -\lambda \overline{\partial T/\partial X}|_{X=0}\,. \tag{13}$$

It is in turn necessary to determine the value of $\bar{T}|_{X=0;\ t=0}$ in order to utilize Eq. (13). As was noted in [6], this requires construction of the interior [11, 12] expansion and splicing with the solution of the exterior problem. Our task is simplified in first approximation by the observation that the interior expansion "works" at shorter time scales [6] than the exterior expansion, where there cannot be any perceptible heat transfer between the system and reservoir. Switching to the "internal" time in relation (11) therefore leads to the conclusion that $\langle T_* \rangle$ is constant, which enables us to find the following expression, analogous to that in [6], by splicing in the (integral) relation in question:

$$\bar{T}|_{x=0;t=0} = \int_0^1 \int_0^1 T_*(x, y)|_{t=0}\, dx dy, \tag{14}$$

where $T_*|_{t=0}$ corresponds to the initial temperature distribution in the reservoir.

Since we have limited ourselves to analyzing the main approximation, we have written expansion formulas for Pe^{-1} of the type of (6), and it should therefore be noted that relations (13)-(14) belong to the zero-order approximation with respect to Pe^{-1} for solution of the problem in the system. The zero index in Eq. (3) can then be omitted [i.e., (3) can be used), and boundary conditions (13)-(14) should, when necessary, be supplemented by any conditions on the remainder of the system boundary that correspond to the situation under analysis.

It is natural to expect that the condition sought will retain a structure similar to that of (13) when the heat reservoir and the interface between it and the system have more complex shapes. If the reservoir boundaries are formed by coordinate surfaces of some orthogonal coordinate system, then the above program can be extended with minor complications to this case by utilizing the effective diffusion (heat conduction) equation obtained in [9]. An analogous approach can be recommended when it is possible to construct a coordinate system from the family of flow functions and lines orthogonal to them. For example, problems of this sort arise in analyzing the internal mass transfer problem for a spherical drop bathed by an exterior flow with small Reynolds numbers, where we arrive at the Kronig–Brink equation within a definite time interval [13, 14]. In these variants, as in those discussed above, the role of the equation of the type of (8) reduced to indication of rapid temperature equalization far from the system boundary as $\mathrm{Pe} \to \infty$, and the average temperature in the reservoir then served as the boundary condition at infinity for the boundary layer equation of the type of (10) in considering the primary approximation for Pe^{-1}. If we postulate a similar role for the average reservoir temperature in more general problems, we can see that the main problem is to establish the feasibility of extending this average parameter to the system–reservoir contact surface, as relation (12) made it possible to do.

Actually, if we utilize the natural coordinates s and n associated with the interface [15], application of the Ostrogradskii–Gauss theorem to the heat conduction equation in the reservoir with no heat fluxes at the outer boundaries (not associated with the system) and the continuity equation taken into account gives us the following expression in place of (11):

$$\rho_* c_* W \frac{d\langle T_* \rangle}{dt} = -\lambda_* \int_S \frac{\partial T_*}{\partial n}\bigg|_{n=0} dS, \quad \langle T_* \rangle = \frac{1}{W} \int_W T_* dW, \tag{15}$$

where W is the reservoir volume and S is the contact surface with the system. By localizing Eqs. (8)-(9) as $\ell_y \to 0$, we can then obtain an approximation equation for the boundary layer:

$$\rho_* c_* \frac{\partial T_*}{\partial t} = \frac{\partial}{\partial n}\left\{[\lambda_* + \varkappa_0(s) n^4] \frac{\partial T_*}{\partial n}\right\}, \tag{16}$$

which generalizes (8) to more complicated situations. Here

$$\varkappa_0(s) = 0.25 \rho_*^2 c_*^2 (\partial v_s / \partial n|_{n=0})^2 / \lambda_* \tag{17}$$

is the flow function at the contact surface. It should be remarked that localization (16) is applicable if there are no places on the contour with small radii of curvature. However, such points make a local contribution to the total energy transfer between the reservoir and the system that is generally negligible in comparison with that made by the heat fluxes through the smooth portions of the contour. We should also note that it is sometimes convenient to associate the quantity $\partial v_s / \partial n$ at the boundary in (17) with the surface friction density.

When (16) is analyzed with boundary conditions similar to (4) taken into account, it is easily ascertained by analogy with the problem examined above that a relation of the type of (12) holds, and this enables us to formulate the "heat reservoir" boundary condition in the form

$$\rho_* c_* W \frac{\partial T}{\partial t}\bigg|_S = -\lambda \int_S \frac{\partial T}{\partial n} dS, \tag{18}$$

where the heat flux on the right side of (18) cannot be removed from under the integral sign in the general case. Both sides of equality (18) contain quantities that depend solely on time.

As before, switching to a shorter time scale (interior expansion) in (15) leads to the conclusion that the quantity $\langle T_* \rangle$ is preserved in the interior solution of the first approximation, which ultimately enables us to supplement (18) with the relation

$$T|_{S;t=0} = \frac{1}{W} \int_W T_*(W,\ 0)\, dW. \tag{19}$$

Conclusions. It must be noted that boundary conditions of the type of (13) and (18) can contain additional terms associated with heat input to the reservoir from outside. However, these fluxes should not be very strong (i.e., should be of appropriate order of magnitude with respect to Pe^{-1}), in order that they can be included in harmonic form in the small-parameter expansions.

The heat dispersion equation obtained with non-one-dimensional liquid flow can be applied to a number of heat and mass transfer problems. Here we have employed it to construct a boundary condition of the "heat reservoir" type by asymptotic methods. The "heat reservoir" condition permits simplified formulation of some multilayer heat and mass transfer problems.

NOTATION

c) specific heat capacity; T) temperature; T_*^i) components of reservoir temperature expansion in parameter ε; t) time; X, Y, x, y) dimensional and dimensionless Cartesian coordinates; λ) coefficient of thermal conductivity; v_s) tangential velocity at system–reservoir interface; ρ) density; $*$) quantities pertaining to reservoir; $\langle\ \rangle$) symbol for averaging.

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